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Note on the thermal history of decoupled massive particles

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Abstract: This note provides an alternative approach to the momentum decay and thermal evolution of decoupled massive particles. Although the ingredients in our results have been addressed in Ref.[1], the strategies employed here are simpler, and the results obtained here are more general.

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1. Introduction

As is well known, for the freely traveling massless particle like photon in an expanding FLRW universe, the frequency or energy will vary inversely proportional to the scale factor, which implies that the number density of massless particles still keeps its thermal spectrum form with a redshifted effective temperature although these particles went out of the thermal equilibrium into the free expansion as time passed. This is the physical foundation for the cosmic microwave radiation background currently observed by us. Now a natural question arises, namely, does the above fact also apply to the massive particle? Not only does this question possess a theoretical interest by itself, but also acquires a practical implication in cosmology since neutrinos and antineutrinos are believed to be massive. However, to my best knowledge, this issue has not been addressed in literatures except in Weinberg's cosmology book published recently[1].

The purpose of this note is to provide an alternative approach to this issue. The strategies employed here are simpler, but the results obtained here are more general. Notations and conventions follow Ref.[2].

2. Momentum Decay

In general curved spacetime, a particle of mass m freely travels along the timelike geodesic $\eta(\tau)$ with τ the proper time, which means that $U^a = (\frac{\partial}{\partial\tau})^a$ gives the geodesic equation

$$U^a\nabla_a U^b = 0 \tag{2.1}$$

with $U^a U_a = -1$. Assume there to be a family of observers Z^a along the geodesic, then we have

$$\frac{dE}{d\tau} = U^a \nabla_a (-m U^b Z_b) = -m U^a U^b \nabla_a Z_b, \tag{2.2}$$

where $E = -mU^b Z_b$ is the energy of massive particle measured by the observers.

Now for the expanding FLRW metric

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{2.3}$$

with $K = 1, 0, -1$ for closed, flat, and open universes respectively, if the observers are chosen to be the isotropic ones as usual, i.e., $Z^a = (\frac{\partial}{\partial t})^a$, we have

$$\nabla_a Z_b = \frac{\dot{a}}{a} h_{ab}, \tag{2.4}$$

where the dot denotes the derivative with respect to the time t, h_{ab} is the induced metric on the surface of constant t, given by $h_{ab} = g_{ab} + (dt)_a (dt)_b$. Plugging Eq.(2.4) into Eq.(2.2), we obtain

$$\frac{dE}{d\tau} = -m\frac{\dot{a}}{a} U^a U^b h_{ab} = -m\frac{\dot{a}}{a}[-1 + (U^a Z_a)^2] = -\frac{E^2 - m^2}{m}\frac{\dot{a}}{a}, \tag{2.5}$$

which implies

$$-\frac{da}{a} = m\frac{dt}{d\tau}\frac{dE}{E^2 - m^2} = \frac{EdE}{E^2 - m^2} = \frac{1}{2}\frac{d(E^2 - m^2)}{E^2 - m^2} = \frac{dp}{p}, \tag{2.6}$$

where $p = \sqrt{E^2 - m^2}$ is the magnitude of momentum of massive particle measured by the isotropic observers. Whence we know that for a freely traveling massive particle in an expanding FLRW universe, it is its momentum rather than energy that goes like[1]

$$p \propto \frac{1}{a}. \tag{2.7}$$

It is noteworthy that this result is also obtained in Ref.[1], where, however, the method employed seems somewhat complicated, and some approximations are also made.

3. Thermal Evolution

Let us assume that during the evolution of our universe, there exists a last scattering surface at the time t_L when some kinds of massive particles such as neutrinos and antineutrinos suddenly went from being in thermal equilibrium to a decoupled expansion.

[1]Of course, the momentum of a massless particle shares the same behavior since its momentum equals energy.

Then according to Eq.(2.7) the massive particle that has momentum p at a later time t would have had momentum $p_L = p\frac{a(t)}{a(t_L)}$ at the time t_L. So the number density of massive particles at the time t with momentum between p and $p + dp$ would be

$$\begin{aligned} n(p,t)dp &= (\frac{a(t_L)}{a(t)})^3 n(p_L, t_L) d(p_L) \\ &= (\frac{a(t_L)}{a(t)})^3 \frac{4\pi g p_L^2 dp_L}{(2\pi\hbar)^3} \frac{1}{\exp[(\sqrt{p_L^2 + m^2} - \mu_d)/kT_d] \pm 1} \\ &= \frac{4\pi g p^2 dp}{(2\pi\hbar)^3} \frac{1}{\exp[(\sqrt{p_L^2 + m^2} - \mu_d)/kT_d] \pm 1} \\ &= \frac{4\pi g p^2 dp}{(2\pi\hbar)^3} \frac{1}{\exp[(\sqrt{p^2 + m_e^2} - \mu_e)/kT_e] \pm 1}. \end{aligned} \tag{3.1}$$

Here the factor $(\frac{a(t_L)}{a(t)})^3$ in the first step arises from the dilution of particles due to the cosmic expansion. The Fermi-Dirac and Bose-Einstein distributions are employed in the second step, where g is the number of spin states of the particle and antiparticles, μ_d and T_d denote the chemical potential and temperature in thermal equilibrium at the last scattering surface, respectively, and the sigh is $+$ for fermions and $-$ for bosons. We introduce the effective mass, chemical potential, and temperature in the last step, i.e., $m_e = m\frac{a(t_L)}{a(t)}, \mu_e = \mu_d\frac{a(t_L)}{a(t)}, T_e = T_d\frac{a(t_L)}{a(t)}$. Therefore the form of the Fermi-Dirac and Bose-Einstein distributions are preserved for the thermal evolution of decoupled massive particle, with the effective mass, chemical potential, and temperature varying inversely proportional to the scale factor a at the same time, which implies that the ratios among the effective mass, chemical potential, and temperature remain constant, just as before decoupling.

Note that just by taking the mass to be zero the above argument obviously reduces to the massless case, where the result is also obtained in Ref.[1] by the thermodynamic method rather than the simpler dynamical picture employed here. In addition, although the spectrum has still kept the form of the Fermi-Dirac and Bose-Einstein distributions since decoupling, it is not the thermal spectrum with the effective temperature and chemical potential since the effective mass is not equal to the static mass. The unique exception is the massless case.

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